

# ABELIAN-BY-CYCLIC MOUFANG LOOPS

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**ABSTRACT.** We use groups with triality to construct a series of nonassociative Moufang loops. Certain members of this series contain an abelian normal subloop with the corresponding quotient being a cyclic group. In particular, we give a new series of examples of finite abelian-by-cyclic Moufang loops. The previously known [10] loops of this type of odd order  $3q^3$ , with prime  $q \equiv 1 \pmod{3}$ , are particular cases of our series. Some of the examples are shown to be embeddable into a Cayley algebra.

**KEYWORDS:** Moufang loops, groups with triality

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## 1. INTRODUCTION

Universal constructions for new Moufang loops are few. An example is Chein's doubling process [2] which allows one, given an arbitrary nonabelian group  $G$ , to obtain a nonassociative Moufang loop of cardinality  $2|G|$ . Since the discovery of the relation between groups with triality and Moufang loops, which has been used successfully by various authors [9, 6, 5] to solve important problems in the theory of Moufang loops, there appeared new ways of constructing Moufang loops using groups with triality.

In the present paper, we build a new series of groups with triality and then derive an explicit multiplication formula for the corresponding Moufang loops. In particular, we obtain a series of *abelian-by-cyclic* Moufang loops (i.e. an upward extension of an abelian group by a cyclic group). To be more precise, let  $R$  be an arbitrary associative commutative unital ring and let  $R_0$  be a cyclic group of invertible elements of  $R$ . We show that the set of tuples  $(r, x, y, z)$ , where  $r \in R_0$ ,  $x, y, z \in R$ , with the multiplication

$$(1) \quad (r_1, x_1, y_1, z_1)(r_2, x_2, y_2, z_2) = (r_1 r_2, x_1 + r_1 x_2, y_1 + r_1 y_2, r_2 z_1 + z_2 + (1 - 2r_1^{-1} r_2) x_1 y_2 - x_2 y_1)$$

is an abelian-by-cyclic nonassociative Moufang loop of the form  $R_0.(R + R + R)$  provided that either  $R_0$  has order 3 or  $R$  has characteristic 2.

The minimal finite loops of this type clearly arise if  $R_0$  has prime order  $p$  and  $R$  is a minimal finite field with an element of multiplicative order  $p$ . For example, this gives abelian-by-cyclic proper Moufang loops of orders  $3 \cdot 2^6$ ,  $7 \cdot 2^9$ ,  $3 \cdot 5^6$ ,  $3 \cdot 7^3$ , etc.

The abelian-by-cyclic Moufang loops are of interest in light of the following problem proposed by M. Kinyon and based on [3]:

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**Problem 1.** *Let  $M$  be a Moufang loop with a normal abelian subgroup (i. e. associative subloop)  $N$  of odd order such that  $M/N$  is a cyclic group of order bigger than 3.*

- (i) *Is  $M$  a group?*
- (ii) *If the orders of  $N$  and  $M/N$  are coprime, is  $M$  a group?*

Although the finite loops of the form (1) are not counterexamples to this problem, there are reasons to believe that they are essentially the only types of abelian-by-cyclic Moufang loops such that the orders of  $N$  and  $M/N$  are coprime.

Examples of abelian-by-cyclic Moufang loops of odd order  $3q^3$  with prime  $q \equiv 1 \pmod{3}$  have also appeared in [10], where the problem of the existence of nonassociative Moufang loops of orders  $pq^3$ , with  $p, q$  prime, was considered. Due to the uniqueness result of [10], these examples must be particular cases of our series (1). However, we have not attempted to find an explicit isomorphism.

The loops (1) are constructed as particular cases of a wider class of Moufang loops  $M_{a,b}$ ,  $a, b \in R$ , not all of which are abelian-by-cyclic but all have the general structure  $R_0.(R+R).R$  for a given subgroup  $R_0 \leq R^\times$ , see Lemma 10. We show that some members of this series can be embedded in the Cayley algebra  $\mathbb{O}(R)$ . In the last section, we raise the isomorphy problem for the loops  $M_{a,b}$  and prove one relevant result.

## 2. PRELIMINARIES

A loop  $M$  is called a *Moufang loop* if  $xy \cdot zx = (x \cdot yz)x$  for all  $x, y, z \in M$ . For basic properties of Moufang loops, see [1].

A group  $G$  possessing automorphisms  $\rho$  and  $\sigma$  that satisfy  $\rho^3 = \sigma^2 = (\rho\sigma)^2 = 1$  is called a *group with triality*  $\langle \rho, \sigma \rangle$  if

$$(x^{-1}x^\sigma)(x^{-1}x^\sigma)^\rho(x^{-1}x^\sigma)^{\rho^2} = 1$$

for every  $x$  in  $G$ . In a group  $G$  with triality  $S = \langle \rho, \sigma \rangle$ , the set  $\mathcal{M}(G) = \{x^{-1}x^\sigma \mid x \in G\}$  is a Moufang loop with respect to the multiplication

$$(2) \quad m.n = m^{-\rho}nm^{-\rho^2}$$

for all  $n, m \in \mathcal{M}(G)$ . Conversely, every Moufang loop arises so from a suitable group with triality. For more information on the relation between groups with triality and Moufang loops, see [7].

Every group with triality  $G$  possesses a (necessarily unique) maximal normal subgroup contained in  $C_G(S)$  which we will denote by  $Z_S(G)$ . For every Moufang loop  $M$  there exists a unique group with triality  $\mathcal{E}(M)$  that satisfies both  $Z_S(\mathcal{E}(M)) = 1$  and  $[\mathcal{E}(M), S] = \mathcal{E}(M)$ .

A homomorphism  $\varphi : G_1 \rightarrow G_2$  of groups  $G_1$  and  $G_2$  with triality  $S$  is called an *S-homomorphism* if  $\alpha\varphi = \varphi\alpha$  for all  $\alpha \in S$ . The following result is a consequence of [4, p. 383–384].

**Lemma 1.** *Moufang loops  $M_1$  and  $M_2$  are isomorphic if and only if  $\mathcal{E}(M_1)$  and  $\mathcal{E}(M_2)$  are  $S$ -isomorphic.*

## 3. TRIALITY REPRESENTATIONS

**Lemma 2.** *Let  $M$  be a Moufang loop. Then, for every  $x, y, m \in M$ , we have*

$$m^{-1}(mx.y) = xm^{-1}.my = (x.ym^{-1}).m$$

*Proof.* This follows from the left and right Moufang identities.  $\square$

**Lemma 3.** *Let  $G$  be a group with triality  $S = \langle \sigma, \rho \rangle$ . Then, for every  $m \in M = \mathcal{M}(G)$ ,  $G$  is a group with triality  $S_{(m)} = \langle \sigma, \rho^2 m \rho^2 \rangle$  which we denote by  $G_{(m)}$ . The Moufang loop  $M_{(m)} = \mathcal{M}(G_{(m)})$  has multiplication*

$$(3) \quad x *_{(m)} y = (x.m^{-1}).(m.y)$$

for all  $x, y \in M_{(m)}$ . In particular,  $M_{(m)}$  is isotopic to  $M$  and, conversely, every loop-isotope of  $M$  has the form  $M_{(m)}$  for some  $m \in M$ .

*Proof.* By the triality identity we have

$$\begin{aligned} (\rho^2 m \rho^2)^3 &= \rho^2 m \rho m \rho m \rho^2 = \rho m^{\rho^2} m^{\rho} m \rho^2 = \rho^3 = 1, \\ (\sigma \rho^2 m \rho^2)^2 &= \rho m^{-1} \rho \rho^2 m \rho^2 = 1. \end{aligned}$$

Hence,  $S_{(m)}$  is an  $S_3$ -complement for  $G$  in  $SG$ . Note that  $M_{(m)}$  coincides with  $M = [G, \sigma]$  as a set. For every  $n \in M_{(m)}$ , we have

$$\begin{aligned} n n^{\rho^2 m \rho^2} n^{(\rho^2 m \rho^2)^2} &= n(m^{-1} n^{\rho^2} m)^{\rho^2} n^{\rho m^{-1} \rho} = \\ &= n m^{-\rho^2} n^{\rho} m^{\rho^2} (m n^{\rho} m^{-1})^{\rho} = n m^{-\rho^2} n^{\rho} (m^{\rho^2} m^{\rho}) n^{\rho^2} m^{-\rho} = \\ &= n m^{-\rho^2} (n^{\rho} m^{-1} n^{\rho^2}) m^{-\rho} = n m^{-\rho^2} (n^{-1}.m^{-1}) m^{-\rho} = \\ &= n((n^{-1}.m^{-1}).m) = n n^{-1} = 1 \end{aligned}$$

Hence,  $G_{(m)}$  is indeed a group with triality  $S_{(m)}$ . The multiplication formula in  $M_{(m)}$  is then given by

$$\begin{aligned} x *_{(m)} y &= x^{-\rho^2 m \rho^2} y x^{\rho m^{-1} \rho} = (m^{-1} x^{-\rho^2} m)^{\rho^2} y (m x^{-\rho} m^{-1})^{\rho} = \\ &= m^{-\rho^2} x^{-\rho} (m^{\rho^2} y m^{\rho}) x^{-\rho^2} m^{-\rho} = m^{-\rho^2} (x^{-\rho} (y.m^{-1}) x^{-\rho^2}) m^{-\rho} = \\ &= m^{-\rho^2} (x.(y.m^{-1})) m^{-\rho} = (x.(y.m^{-1})).m = (x.m^{-1}).(m.y) \end{aligned}$$

where the last equality holds by Lemma 2.

By [1, Lemma VII.5.8] every loop-isotope of a Moufang loop is isomorphic to a principal isotope with multiplication of the form (3).  $\square$

Let  $R$  be a commutative ring and  $S = \langle \sigma, \rho \rangle$ . A right  $RS$ -module  $V$  is called a *triality module (for  $S$ )* if  $V$  is a group with triality  $S$ . This holds if and only if  $(\sigma - 1)(1 + \rho + \rho^2)$  annihilates  $V$ . The classification of the indecomposable triality  $RS$ -modules over fields is given in [8, Lemma 5].

Let  $G$  be a group with triality  $S$ . A right  $R[S \ltimes G]$ -module  $V$  is called a *triality module (for  $G$ )* if the natural semidirect product  $G \ltimes V$  is a group with triality  $S$ .

**Lemma 4.** *An  $R[S \ltimes G]$ -module  $V$  is a triality module for  $G$  if and only if the restriction  $V|_{S_{(m)}}$  is a triality module for  $S_{(m)}$  for every  $m \in \mathcal{M}(G)$ .*

*Proof.* If  $V$  is a triality module for  $G$  then  $V|_{S_{(m)}}$  is a triality module for  $S_{(m)}$  by Lemma 3, since  $\mathcal{M}(G) \leq \mathcal{M}(G \ltimes V)$ . Let us prove the converse.

Let  $g \in G$  and  $v \in V$ . Set  $x = gv \in G \ltimes V$ . We have

$$[x, \sigma] = v^{-1} m v^{\sigma} = m v^{-m+\sigma},$$

where  $m = [g, \sigma]$ . Since  $mm^\rho m^{\rho^2} = 1$ , we obtain

$$\begin{aligned} [x, \sigma][x, \sigma]^\rho [x, \sigma]^{\rho^2} &= mv^{-m+\sigma} m^\rho v^{-m\rho+\sigma\rho} m^{\rho^2} v^{-m\rho^2+\sigma\rho^2} = \\ mm^\rho m^{\rho^2} v^{-mm^\rho m^{\rho^2} + \sigma m^\rho m^{\rho^2} - m\rho m^{\rho^2} + \sigma\rho m^{\rho^2} - m\rho^2 + \sigma\rho^2} &= \\ v^{-1+\sigma m^{-1} - \rho m^{-1} + \sigma\rho^2 m^{\rho^2} - m\rho^2 + \sigma\rho^2} &= v^{(\sigma-\rho)(m^{-1} + \rho^2 m^{\rho^2} + \rho^2)} = \\ v^{\rho^2(\sigma-1)(\rho^2 m^{-1} \rho^2 + \rho m\rho + 1)\rho} &= 0, \end{aligned}$$

where the last equality holds because the operator  $(\sigma - 1)(\rho^2 m^{-1} \rho^2 + \rho m\rho + 1)$  annihilates  $V$  by the assumption. The claim follows.  $\square$

#### 4. TRIALITY MODULES AND TENSOR PRODUCT

Let  $H$  be a group with triality  $S$ . Denote  $\tilde{H} = S \ltimes H$ . Let  $V_1, V_2, U$  be triality  $R\tilde{H}$ -modules. Suppose that  $\varphi : V_1 \otimes V_2 \rightarrow U$  is an  $R\tilde{H}$ -module homomorphism. For brevity, we will write  $v_1 \boxtimes v_2 = \varphi(v_1 \otimes v_2)$ , where  $v_1 \in V_1, v_2 \in V_2$ . Then, in particular, we have

$$(4) \quad v_1^h \boxtimes v_2^h = (v_1 \boxtimes v_2)^h$$

for all  $h \in \tilde{H}$ . We endow the Cartesian product  $W = V_1 \times V_2 \times U$  with the operation

$$(5) \quad (v_1, v_2, u)(v'_1, v'_2, u') = (v_1 + v'_1, v_2 + v'_2, u + u' + v_1 \boxtimes v'_2)$$

which turns  $W$  into a nilpotent group of class (at most) 2 with a central subgroup (isomorphic to)  $U$ . Moreover, setting  $(v_1, v_2, u)^h = (v_1^h, v_2^h, u^h)$  for every  $h \in \tilde{H}$  and  $(v_1, v_2, u) \in W$  (with the  $R\tilde{H}$ -module action of  $h$  on the components) correctly defines an action of  $\tilde{H}$  on  $W$  due to (4). The resulting group  $G = H \ltimes W$  has a natural  $S$ -action, and the normal series

$$1 \trianglelefteq U \trianglelefteq W \trianglelefteq G$$

is  $S$ -invariant with the corresponding factors being groups with triality  $S$ . In general, the triality on the factors of a normal series of a group does not imply the triality on the whole group. We obtain the following criterion:

**Lemma 5.** *The group  $G$  constructed above is a group with triality  $S$  if and only if*

$$l_1^{\rho^2 m \rho^2} \boxtimes l_2^{(\rho^2 m \rho^2)^2} \in C_U(\sigma)$$

for all  $m \in \mathcal{M}(H)$ ,  $l_1 \in \mathcal{M}(V_1)$ ,  $l_2 \in \mathcal{M}(V_2)$ .

*Proof.* Elements of  $G$  will be written as  $(h, v_1, v_2, u)$ , where  $h \in H$ ,  $(v_1, v_2, u) \in W$ . Then the multiplication and inversion in  $G$  are given explicitly by

$$(6) \quad \begin{aligned} (h, v_1, v_2, u)(h', v'_1, v'_2, u') &= (hh', v_1^{h'} + v'_1, v_2^{h'} + v'_2, u^{h'} + u' + v_1^{h'} \boxtimes v'_2), \\ (h, v_1, v_2, u)^{-1} &= (h^{-1}, -v_1^{h^{-1}}, -v_2^{h^{-1}}, -u^{h^{-1}} + v_1^{h^{-1}} \boxtimes v_2^{h^{-1}}). \end{aligned}$$

We now check the triality for  $G$ . Let  $g = (h, v_1, v_2, u) \in G$ . Then setting  $m = h^{-1}h^\sigma$  we have

$$(7) \quad g^{-1}g^\sigma = (m, v_1^{-m+\sigma}, v_2^{-m+\sigma}, u^{-m+\sigma} + v_1^m \boxtimes v_2^{m-\sigma}).$$

Using the fact that  $H$  is a group with triality and that  $V_1, V_2, U$  are triality  $R\tilde{H}$ -modules we have

$$\begin{aligned}
& (g^{-1}g^\sigma)(g^{-1}g^\sigma)^\rho(g^{-1}g^\sigma)^{\rho^2} \\
&= (m, v_1^{-m+\sigma}, v_2^{-m+\sigma}, u^{-m+\sigma} + v_1^m \boxtimes v_2^{m-\sigma}) \\
&\times (m^\rho, v_1^{-m\rho+\sigma\rho}, v_2^{-m\rho+\sigma\rho}, u^{-m\rho+\sigma\rho} + v_1^{m\rho} \boxtimes v_2^{m\rho-\sigma\rho}) \\
&\times (m^{\rho^2}, v_1^{-m\rho^2+\sigma\rho^2}, v_2^{-m\rho^2+\sigma\rho^2}, u^{-m\rho^2+\sigma\rho^2} + v_1^{m\rho^2} \boxtimes v_2^{m\rho^2-\sigma\rho^2}) \\
&= (mm^\rho, v_1^{-mm^\rho+\sigma m^\rho-m\rho+\sigma\rho}, v_2^{-mm^\rho+\sigma m^\rho-m\rho+\sigma\rho}, u^{-mm^\rho+\sigma m^\rho-m\rho+\sigma\rho} \\
&\quad + v_1^{mm^\rho} \boxtimes v_2^{mm^\rho-\sigma m^\rho} + v_1^{m\rho} \boxtimes v_2^{m\rho-\sigma\rho} + v_1^{-mm^\rho+\sigma m^\rho} \boxtimes v_2^{-m\rho+\sigma\rho}) \\
&\times (m^{\rho^2}, v_1^{-m\rho^2+\sigma\rho^2}, v_2^{-m\rho^2+\sigma\rho^2}, u^{-m\rho^2+\sigma\rho^2} + v_1^{m\rho^2} \boxtimes v_2^{m\rho^2-\sigma\rho^2}) =
\end{aligned}$$

[by  $mm^\rho m^{\rho^2} = 1$ ]

$$\begin{aligned}
&= (1, v_1^{-1+\sigma m^{-1}-\rho m^{-1}+\sigma\rho^2 m\rho^2-m\rho^2+\sigma\rho^2}, v_2^{-1+\sigma m^{-1}-\rho m^{-1}+\sigma\rho^2 m\rho^2-m\rho^2+\sigma\rho^2}, \\
&\quad u^{-1+\sigma m^{-1}-\rho m^{-1}+\sigma\rho^2 m\rho^2-m\rho^2+\sigma\rho^2} + v_1 \boxtimes v_2^{1-\sigma m^{-1}} + v_1^{\rho m^{-1}} \boxtimes v_2^{\rho m^{-1}-\sigma\rho^2 m\rho^2} \\
&\quad + v_1^{-1+\sigma m^{-1}} \boxtimes v_2^{-\rho m^{-1}+\sigma\rho^2 m\rho^2} + v_1^{m\rho^2} \boxtimes v_2^{m\rho^2-\sigma\rho^2} \\
&\quad + v_1^{-1+\sigma m^{-1}-\rho m^{-1}+\sigma\rho^2 m\rho^2} \boxtimes v_2^{-m\rho^2+\sigma\rho^2}) =
\end{aligned}$$

[by  $w^{-1+\sigma m^{-1}-\rho m^{-1}+\sigma\rho^2 m\rho^2-m\rho^2+\sigma\rho^2} = 0$  for  $w = v_1, v_2, u$  (as in proof of Lemma 4)]

$$\begin{aligned}
&= (1, 0, 0, v_1 \boxtimes v_2^{1-\sigma m^{-1}+\rho m^{-1}-\sigma\rho^2 m\rho^2} + v_1^{\rho m^{-1}} \boxtimes v_2^{\rho m^{-1}-\sigma\rho^2 m\rho^2} \\
&\quad + v_1^{\sigma m^{-1}} \boxtimes v_2^{-\rho m^{-1}+\sigma\rho^2 m\rho^2} + v_1^{m\rho^2} \boxtimes v_2^{m\rho^2-\sigma\rho^2} + v_1^{m\rho^2-\sigma\rho^2} \boxtimes v_2^{-m\rho^2+\sigma\rho^2}) \\
&= (1, 0, 0, v_1 \boxtimes v_2^{-m\rho^2+\sigma\rho^2} + v_1^{\rho m^{-1}-\sigma m^{-1}} \boxtimes v_2^{\rho m^{-1}-\sigma\rho^2 m\rho^2} + v_1^{-\sigma\rho^2} \boxtimes v_2^{-m\rho^2+\sigma\rho^2}) \\
&= (1, 0, 0, -v_1^{\rho^2(1-\sigma)\rho} \boxtimes v_2^{\rho^2 m^{-1}\rho^2(1-\sigma)\rho^2 m^{-1}} + v_1^{\rho^2(1-\sigma)\rho^2 m^{-1}} \boxtimes v_2^{\rho^2 m^{-1}\rho^2(1-\sigma)\rho}).
\end{aligned}$$

The elements  $l_1 = v_1^{\rho^2(1-\sigma)}$  and  $l_2 = v_2^{\rho^2 m^{-1}\rho^2(1-\sigma)}$  run through  $\mathcal{M}(V_1)$  and  $\mathcal{M}(V_2)$  as  $v_1$  and  $v_2$  run through  $V_1$  and  $V_2$ , respectively. Hence,  $G$  is a group with triality if and only if the element

$$-l_1^\rho \boxtimes l_2^{\rho^2 m^{-1}} + l_1^{\rho^2 m^{-1}} \boxtimes l_2^\rho = (-l + l^\sigma)^{\rho m\rho^2}$$

is zero, where  $l = l_1^{\rho^2 m^{-1}\rho^2} \boxtimes l_2^{(\rho^2 m^{-1}\rho^2)^2}$ . The claim follows.  $\square$

## 5. THE GROUP WITH TRIALITY

Let  $R$  be a commutative unital ring and let  $R_0$  be a subgroup of  $R^\times$ . We set  $T = R_0 \times R_0$  and let  $S = \langle \sigma, \rho \rangle$  act on  $T$  according to

$$[\sigma] = \begin{pmatrix} -1 & \cdot \\ 1 & 1 \end{pmatrix}, \quad [\rho] = \begin{pmatrix} \cdot & 1 \\ -1 & -1 \end{pmatrix},$$

e.g.,  $(r_1, r_2)^\sigma = (r_1^{-1}r_2, r_2)$ ,  $r_i \in R_0$ . Then  $T$  is a group with triality  $S$  with  $\mathcal{M}(T) = \{(r, 1) \mid r \in R_0\} \cong R_0$ . Denote  $\tilde{T} = S \ltimes T$ .

We define an  $R\tilde{T}$ -module  $V$  as a free  $R$ -module of rank 3 with basis  $e = \{e_1, e_2, e_3\}$  in which the corresponding  $R$ -representation  $\Psi$  for  $\tilde{T}$  is given by

$$(8) \quad \begin{aligned} \Psi : (r_1, r_2) &\mapsto \begin{pmatrix} r_1 & \cdot & \cdot \\ \cdot & r_1^{-1}r_2 & \cdot \\ \cdot & \cdot & r_2^{-1} \end{pmatrix}, \\ \sigma &\mapsto \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}, \quad \rho \mapsto \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \end{pmatrix}. \end{aligned}$$

Swapping  $r \leftrightarrow r^{-1}$  for  $r \in R_0$  gives the matrices of the contragredient representation  $\Psi^*$  in the dual basis  $e^* = \{e_1^*, e_2^*, e_3^*\}$  of the corresponding module  $V^*$ .

**Lemma 6.** *The  $R\tilde{T}$ -modules  $V$  and  $V^*$  are triality modules for  $T$ .*

*Proof.* By duality, we may consider  $V$  only. A direct verification shows that  $\Psi((\sigma - 1)(1 + \tau + \tau^2))$  is the zero matrix, where  $\tau$  has the form

$$(9) \quad \rho^2 m \rho^2 = \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & r \\ r^{-1} & \cdot & \cdot \end{pmatrix}$$

for  $m = (r, 1) \in \mathcal{M}(T)$ ,  $r \in R_0$ . The claim now follows by Lemma 4.  $\square$

The contragredient module  $V^*$  can be realized as a direct summand of the symmetric square  $S^2V$  due to the following

**Lemma 7.** *The 2-homogeneous component of  $R[x_1, x_2, x_3]$  splits under the action of  $\tilde{T}$  given by  $t : x_i \mapsto \sum_j (\Psi(t))_{ij} x_j$ ,  $t \in \tilde{T}$  into the direct sum*

$$(10) \quad \langle x_2 x_3, x_1 x_3, x_1 x_2 \rangle_R \oplus \langle x_1^2, x_2^2, x_3^2 \rangle_R.$$

*The first summand is isomorphic to  $V^*$  as an  $R\tilde{T}$ -module under the map  $\gamma : x_i x_j \mapsto e_k^*$  whenever  $\{i, j, k\} = \{1, 2, 3\}$ . The second summand is isomorphic to  $V^*$  under  $\delta : x_i^2 \mapsto e_i^*$  for  $i = 1, 2, 3$  provided that  $R_0$  has exponent 3.*

*Proof.* For every  $t \in \tilde{T}$ , the matrix  $\Psi(t)$  is monomial, hence has the form  $\sum_k r_k e_{k, k\tau}$  for suitable  $r_k \in R$  and  $\tau \in \text{Sym}_3$ , where  $e_{i, j}$  are the matrix units. Therefore,  $t$  acts by  $t : x_i \mapsto r_i x_{i\tau}$ , which implies the decomposition (10). It also implies that

$$t : x_i x_j \mapsto r_i r_j x_{i\tau, j\tau} \xrightarrow{\gamma} r_k^{-1} e_{k\tau}^*$$

whenever  $\{i, j, k\} = \{1, 2, 3\}$ , because for all matrices in (8) we have  $r_1 r_2 r_3 = 1$ ; and

$$t : x_i^2 \mapsto r_i^2 x_{i\tau}^2 \xrightarrow{\delta} r_i^{-1} e_{i\tau}^*,$$

if  $r_i^3 = 1$ . However,  $\Psi^*(t) = \sum_k r_k^{-1} e_{k, k\tau}$ . Hence,  $\gamma$  extends to an isomorphism onto  $V^*$  and so does  $\delta$  whenever  $R_0$  has exponent 3.  $\square$

Let  $a, b \in R$  be fixed elements at least one of which is invertible. We henceforth assume that one of the following conditions is fulfilled:

- (I)  $R_0^3 = 1$ ,
- (II)  $b = 0$ .

Then by Lemma 6 the submodule of  $S^2V$  spanned by  $ax_i^2 + bx_jx_k$ , where  $\{i, j, k\} = \{1, 2, 3\}$  is isomorphic to  $V^*$ . The invertibility of either of  $a, b$  ensures that this submodule is complemented in  $S^2V$ . Hence, there is an  $R\tilde{T}$ -module homomorphism  $\varphi_{a,b} : V \otimes V \rightarrow V^*$  which is written in the bases  $\mathbf{e}$  and  $\mathbf{e}^*$  as

$$(11) \quad \begin{aligned} (v_1, v_2, v_3) \boxtimes (w_1, w_2, w_3) = & (a(v_2w_3 + v_3w_2) + bv_1w_1, \\ & a(v_1w_3 + v_3w_1) + bv_2w_2, a(v_1w_2 + v_2w_1) + bv_3w_3), \end{aligned}$$

where  $\boxtimes = \boxtimes_{a,b} = \varphi_{a,b} \circ \otimes$ . By the discussion in Section 4, we may construct the group  $G = T \ltimes W$ , where  $W = V \times V \times V^*$  has the operation (11). We will henceforth denote  $\tilde{G} = \tilde{T} \ltimes W = S \ltimes G$ .

**Lemma 8.** *We have*

- (i)  $G$  is a group with triality  $S$ ,
- (ii)  $Z_S(G) = 1$  and  $[G, S] = G$ .

*Proof.* (i) Let  $l_1, l_2$  be arbitrary elements of  $\mathcal{M}(V)$ . Then there exist  $s_1, s_2 \in R$  such that  $l_i = (s_i, -s_i, 0)$ . Let  $m \in \mathcal{M}(T)$ . Then  $m = (r, 1)$  for some  $r \in R_0$  and  $\rho^2 m \rho^2$  is as in (9). By (11), we have

$$l_1^{\rho^2 m \rho^2} \boxtimes l_2^{(\rho^2 m \rho^2)^2} = (0, s_1, -s_1 r) \boxtimes (-s_2, 0, r s_2) = s_1 s_2 (ar, ar, -a - br^2),$$

which lies in  $C_{V^*}(\sigma)$ . The claim follows by Lemma 5.

(ii) Clearly, every proper nontrivial normal  $S$ -invariant subgroup of  $G$  must include  $V^*$  and be included in  $W$ . Since  $S$  induces a nontrivial action on both  $V^*$  and  $G/W$ , the claim follows. □

## 6. THE MOUFANG LOOP

Lemma 8 implies the existence of a Moufang loop  $\mathcal{M}(G)$  which depends on the parameters  $R, R_0, a, b$ . Assuming that  $R$  and  $R_0$  are fixed, we will denote this loop by  $M_{a,b}$  and determine its structure.

**Lemma 9.** *The Moufang loop  $M_{a,b}$  consists of the elements of  $G$  of the form*

$$(12) \quad ((r, 1), x(1, -r^{-1}, 0), y(1, -r^{-1}, 0), z(-r^{-1}, 1, 0) + xy(b, 0, -ar^{-1})).$$

for  $r \in R_0$ ,  $x, y, z \in R$ .

*Proof.* Let  $g = (t, v_1, v_2, u) \in G$ . Then  $g^{-1}g^\sigma$  is given by (7), where  $m = t^{-1}t^\sigma = (1, r) \in \mathcal{M}(T)$ ,  $v_i = (v_{i1}, v_{i2}, v_{i3})$ ,  $i = 1, 2$ ,  $u = (u_1, u_2, u_3)$ . Due to

$$\Psi(-m + \sigma) = \begin{pmatrix} -r & 1 & . \\ 1 & -r^{-1} & . \\ . & . & . \end{pmatrix},$$

we have

$$\begin{aligned} v_1^{-m+\sigma} &= v_1 \Psi(-m + \sigma) = x(1, -r^{-1}, 0), \quad \text{where } x = -rv_{11} + v_{12}, \\ v_2^{-m+\sigma} &= v_2 \Psi(-m + \sigma) = y(1, -r^{-1}, 1, 0), \quad \text{where } y = -rv_{21} + v_{22}, \\ u^{-m+\sigma} &= v_i \Psi^*(-m + \sigma) = w(-r^{-1}, 1, 0), \quad \text{where } w = u_1 - ru_2, \\ v_1^m \boxtimes v_2^{m-\sigma} &= (rv_{11}, r^{-1}v_{12}, v_{13}) \boxtimes (-y, r^{-1}y, 0) = (z_1, z_2, -ar^{-1}xy), \quad \text{where} \\ & z_1 = y(ar^{-1}v_{13} - brv_{11}), \quad z_2 = y(-av_{13} + br^{-2}v_{12}). \end{aligned}$$

Since we assume (I) or (II), we have  $br^3 = b$  and so

$$z_1 + r^{-1}z_2 = y(-brv_{11} + br^{-3}v_{12}) = bxy.$$

Hence, setting  $z = w + z_2$ , we have the required form of  $g^{-1}g^\sigma$ .  $\square$

By Lemma 9, sending an element (12) to the tuple

$$(r, x, y, z) \quad r \in R_0, \quad x, y, z \in R$$

gives a bijection from  $M_{a,b}$ . We will therefore assume that  $M_{a,b}$  consists of all such tuples.

**Lemma 10.** *The multiplication and inversion in the Moufang loop  $M_{a,b}$  are given by*

$$\begin{aligned} (r_1, x_1, y_1, z_1)(r_2, x_2, y_2, z_2) = \\ (r_1r_2, x_1 + r_1x_2, y_1 + r_1y_2, r_2z_1 + z_2 + a(x_1y_2 - x_2y_1) + br_1^{-1}r_2x_1y_2), \\ (r, x, y, z)^{-1} = (r^{-1}, -r^{-1}x, -r^{-1}y, -r^{-1}z + bxy). \end{aligned}$$

*Proof.* We first derive the inversion formula using (6). Denoting  $m = (r, 1)$  we have

$$\begin{aligned} (r, x, y, z)^{-1} &\leftrightarrow (m, x(1, -r^{-1}, 0), y(1, -r^{-1}, 0), z(-r^{-1}, 1, 0) + xy(b, 0, -ar^{-1}))^{-1} \\ &= (m^{-1}, -x(1, -r^{-1}, 0)^{m^{-1}}, -y(1, -r^{-1}, 0)^{m^{-1}}, -z(-r^{-1}, 1, 0)^{m^{-1}} \\ &\quad - xy(b, 0, -ar^{-1})^{m^{-1}} + x(1, -r^{-1}, 0)^{m^{-1}} \boxtimes y(1, -r^{-1}, 0)^{m^{-1}}) \\ &= (m^{-1}, -x(r^{-1}, -1, 0), -y(r^{-1}, -1, 0), -z(-1, r^{-1}, 0) - xy(br, 0, -ar^{-1}) \\ &\quad + xy(br^{-2}, b, -2ar^{-1})) = ((r^{-1}, 1), -r^{-1}x(1, -r, 0), -r^{-1}y(1, -r, 0), \\ &\quad - r^{-1}z(-r, 1, 0) + bxy(-r, 1, 0) + (-r^{-1}x)(-r^{-1}y)(b, 0, -ar)) \\ &\leftrightarrow (r^{-1}, -r^{-1}x, -r^{-1}y, -r^{-1}z + bxy). \end{aligned}$$

Similarly, with  $m_i = (r_i, 1)$ ,  $i = 1, 2$ , we have

$$\begin{aligned} (r_1, x_1, y_1, z_1)(r_2, x_2, y_2, z_2) &\leftrightarrow (m_1, x_1(1, -r_1^{-1}, 0), y_1(1, -r_1^{-1}, 0), z_1(-r_1^{-1}, 1, 0) \\ &\quad + x_1y_1(b, 0, -ar_1^{-1}))^{-\rho} (m_2, x_2(1, -r_2^{-1}, 0), y_2(1, -r_2^{-1}, 0), z_2(-r_2^{-1}, 1, 0) \\ &\quad + x_2y_2(b, 0, -ar_2^{-1})) (m_1, x_1(1, -r_1^{-1}, 0), y_1(1, -r_1^{-1}, 0), z_1(-r_1^{-1}, 1, 0) \\ &\quad + x_1y_1(b, 0, -ar_1^{-1}))^{-\rho^2} = (m_1^{-\rho}, x_1(-r_1^{-1}, 1, 0)^\rho, y_1(-r_1^{-1}, 1, 0)^\rho, z_1(1, -r_1^{-1}, 0)^\rho \\ &\quad + x_1y_1(0, b, -ar_1^{-1})^\rho) (m_2, x_2(1, -r_2^{-1}, 0), y_2(1, -r_2^{-1}, 0), z_2(-r_2^{-1}, 1, 0) \\ &\quad + x_2y_2(0, b, -ar_2^{-1})) (m_1^{-\rho^2}, x_1(1, -r_1, 0)^{\rho^2}, y_1(1, -r_1, 0)^{\rho^2}, z_1(-r_1, 1, 0)^{\rho^2} \\ &\quad + x_1y_1(0, b, -ar_1^{-1})^{\rho^2}) = ((r_2, r_1^{-1}), x_1(0, -r_1^{-1}, 1)^{m_2} + x_2(1, -r_2^{-1}, 0), \\ &\quad y_1(0, -r_1^{-1}, 1)^{m_2} + y_2(1, -r_2^{-1}, 0), z_1(0, 1, -r_1^{-1})^{m_2} + z_2(-r_2^{-1}, 1, 0) \\ &\quad + x_1y_1(-ar_1^{-1}, 0, b)^{m_2} + x_2y_2(b, 0, -ar_2^{-1}) + x_1y_2(0, -r_1^{-1}, 1)^{m_2} \boxtimes (1, -r_2^{-1}, 0)) \\ &\times ((r_1, r_1), x_1(1, 0, -r_1^{-1}), y_1(1, 0, -r_1^{-1}), z_1(-r_1^{-1}, 0, 1) + x_1y_1(b, -ar_1^{-1}, 0)) \end{aligned}$$



$$\begin{aligned}
&= ((r_1 r_2, 1), (x_2, -r_1^{-1} r_2^{-1} x_1 - r_2^{-1} x_2, x_1))^{m_1^{-\rho^2}} + x_1(1, 0, -r_1^{-1}), \\
&\quad (y_2, -r_1^{-1} r_2^{-1} y_1 - r_2^{-1} y_2, y_1))^{m_1^{-\rho^2}} + y_1(1, 0, -r_1^{-1}), (-r_2^{-1} z_2, r_2 z_1 + z_2, -r_1^{-1} z_1))^{m_1^{-\rho^2}} \\
&\quad + z_1(-r_1^{-1}, 0, 1) + (-ar_1^{-1} r_2^{-1} x_1 y_1 + bx_2 y_2, 0, bx_1 y_1 - ar_2^{-1} x_2 y_2))^{m_1^{-\rho^2}} \\
&\quad + x_1 y_1(b, -ar_1^{-1}, 0) + x_1 y_2(-ar_2^{-1}, a + br_1^{-1} r_2, -ar_1^{-1} r_2^{-1})^{m_1^{-\rho^2}} \\
&\quad + (x_2, -r_1^{-1} r_2^{-1} x_1 - r_2^{-1} x_2, x_1))^{m_1^{-\rho^2}} \boxtimes y_1(1, 0, -r_1^{-1})) \\
&= ((r_1 r_2, 1), (x_1 + r_1 x_2)(1, -r_1^{-1} r_2^{-1}, 0), (y_1 + r_1 y_2)(1, -r_1^{-1} r_2^{-1}, 0), \\
&\quad (r_2 z_1 + z_2)(-r_1^{-1} r_2^{-1}, 1, 0) + (-ar_1^{-2} r_2^{-1} x_1 y_1 + br_1^{-1} x_2 y_2, 0, br_1 x_1 y_1 - ar_1 r_2^{-1} x_2 y_2) \\
&\quad + x_1 y_1(b, -ar_1^{-1}, 0) + x_1 y_2(-ar_1^{-1} r_2^{-1}, a + br_1^{-1} r_2, -ar_2^{-1}) \\
&\quad + (a(r_1^{-2} r_2^{-1} x_1 y_1 + r_1^{-1} r_2^{-1} x_2 y_1) + br_1 x_2 y_1, a(-x_2 y_1 + r_1^{-1} x_1 y_1), \\
&\quad a(-r_1^{-1} r_2^{-1} x_1 y_1 - r_2^{-1} x_2 y_1) - br_1^{-2} x_1 y_1)) = ((r_1 r_2, 1), (x_1 + r_1 x_2)(1, -r_1^{-1} r_2^{-1}, 0), \\
&\quad (y_1 + r_1 y_2)(1, -r_1^{-1} r_2^{-1}, 0), (r_2 z_1 + z_2 + a(x_1 y_2 - x_2 y_1) + br_1^{-1} r_2 x_1 y_2)(-r_1^{-1} r_2^{-1}, 1, 0) \\
&\quad + (x_1 + r_1 x_2)(y_1 + r_1 y_2)(b, 0, -ar_1^{-1} r_2^{-1})) \leftrightarrow (r_1 r_2, x_1 + r_1 x_2, y_1 + r_1 y_2, \\
&\quad r_2 z_1 + z_2 + a(x_1 y_2 - x_2 y_1) + br_1^{-1} r_2 x_1 y_2).
\end{aligned}$$

□

By Lemma 10, the loop  $M_{a,b}$  has the form  $R_0.N$ , where  $N = (R + R).R$  is a normal subgroup. It can be checked that the subgroup  $N$  is commutative if and only if  $2a + b = 0$ . The associator of  $(r_i, x_i, y_i, z_i) \in M_{a,b}$ ,  $i = 1, 2, 3$ , is trivial if and only if

$$a \begin{vmatrix} r_1 - 1 & r_2 - 1 & r_3 - 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 0.$$

In particular,  $M_{a,b}$  is nonassociative, if  $a \neq 0$ ,  $|R_0| > 1$ , and  $R$  is a domain, in which case  $\text{Nuc}(M_{a,b})$  consists of the elements  $(1, 0, 0, z)$ ,  $z \in R$ .

We observe that, for every  $c \in R^\times$ , the loops  $M_{a,b}$  and  $M_{ca,cb}$  are isomorphic, which can be shown by changing the "coordinates"  $(r, x, y, z) \mapsto (r, x, y, cz)$  in  $M_{a,b}$ . Hence, we may only consider the loops  $M_{1,b}$  and  $M_{a,1}$ . In particular, we obtain abelian-by-cyclic Moufang loops,  $M_{1,-2}$ , with the multiplication (1) which split into two types:

- (I) if the characteristic of  $R$  is not 2 then  $M_{1,-2}$  has the form  $3.(R + R + R)$ , i.e.  $R_0$  is cyclic of order 3,
- (II) if the characteristic of  $R$  is 2 then  $M_{1,-2} = M_{1,0}$  has the form  $R_0.N$ , where  $R_0$  is an arbitrary cyclic subgroup of  $R^\times$  and  $N = R + R + R$  is an elementary abelian 2-group.

## 7. EMBEDDING IN THE CAYLEY ALGEBRA

We show that a particular case of the above-constructed series of Moufang loops, namely  $M_{1,0}$ , can be embedded in a Cayley algebra. Recall that the Cayley algebra  $\mathbb{O} = \mathbb{O}(R)$  can be defined as set of all *Zorn matrices*

$$\begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix}, \quad a, b \in R, \quad \mathbf{v}, \mathbf{w} \in R^3$$

with the natural structure of a free  $R$ -module and multiplication given by the rule

$$(13) \quad \begin{pmatrix} a_1 & \mathbf{v}_1 \\ \mathbf{w}_1 & b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & \mathbf{v}_2 \\ \mathbf{w}_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + \mathbf{v}_1 \cdot \mathbf{w}_2 & a_1 \mathbf{v}_2 + b_2 \mathbf{v}_1 \\ a_2 \mathbf{w}_1 + b_1 \mathbf{w}_2 & \mathbf{w}_1 \cdot \mathbf{v}_2 + b_1 b_2 \end{pmatrix} + \begin{pmatrix} 0 & -\mathbf{w}_1 \times \mathbf{w}_2 \\ \mathbf{v}_1 \times \mathbf{v}_2 & 0 \end{pmatrix},$$

where, for  $\mathbf{v} = (v_1, v_2, v_3)$  and  $\mathbf{w} = (w_1, w_2, w_3)$  in  $R^3$ , we denoted

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= v_1 w_1 + v_2 w_2 + v_3 w_3 \in R, \\ \mathbf{v} \times \mathbf{w} &= (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1) \in R^3. \end{aligned}$$

It is well-known that  $\mathbb{O}$  is an alternative algebra and the set of its invertible elements  $\mathbb{O}^\times$  forms a Moufang loop.

Consider the subset of  $\mathbb{O}^\times$  of elements of the form

$$\begin{pmatrix} r & (0, x, y) \\ (z, 0, 0) & 1 \end{pmatrix}$$

which we identify with the tuples  $(r, x, y, z)$ , where  $r \in R_0 \leq R^\times$ ,  $x, y, z \in R$ . Using (13) it can be checked that this subset is a subloop with the multiplication

$$\begin{aligned} (r_1, x_1, y_1, z_1)(r_2, x_2, y_2, z_2) &= \\ (r_1 r_2, x_1 + r_1 x_2, y_1 + r_1 y_2, r_2 z_1 + z_2 + x_1 y_2 - x_2 y_1), \end{aligned}$$

hence is isomorphic to  $M_{1,0}$ .

## 8. THE ISOMORPHY PROBLEM

Clearly, the loops  $M_{a,b}$  and  $M_{a',b'}$  are isomorphic if  $(a', b') = (a, b)^\varphi$  for some  $\varphi \in \text{Aut}(R)$ . However, determining all isomorphisms among the loops  $M_{a,b}$  seems to be a challenging problem. In particular, we state

**Problem 2.** *Can the loops  $M_{1,b}$  and  $M_{1,b'}$  be isomorphic for non- $\text{Aut}(R)$ -conjugate elements  $b, b' \in R$ ?*

We only prove the following particular result.

**Proposition 11.** *Let  $R$  be a field and let  $0 \neq b \in R$ . Then  $M_{1,0}$  is not isomorphic to  $M_{1,b}$ .*

*Proof.* By lemma 8(ii), the above-constructed group with triality  $G = G_{a,b}$  coincides with  $\mathcal{E}(M_{a,b})$ . By Lemma 1, it suffices to show that  $G_{1,0}$  is not isomorphic to  $G_{1,b}$ . Observe that, since  $R_0^3 = 1$  and  $R$  is a field, we have  $|R_0| = 3$ . By the construction of  $G$ , we have  $W = [G, G]$ ,  $V^* = Z(W)$ , and  $W/V^* \cong V \oplus V$  is the direct sum of three  $RT$ -homogeneous 2-dimensional components  $Q_i$ ,  $i = 1, 2, 3$ , spanned by the pairs  $(e_i, 0)$  and  $(0, e_i)$ , where  $e_i$  is the  $i$ th basis vector for  $V$ . By (6) and (11), the full preimage  $\widehat{Q}_i$  of  $Q_i$  in  $W$  can be identified with a group with the multiplication

$$(r_1, r_2, u)(r'_1, r'_2, u') = (r_1 + r'_1, r_2 + r'_2, u + u' + br_1 r'_2 e_i^*),$$

where  $r_j, r'_j \in R$  and  $u, u' \in V^*$ . Hence, the groups  $\widehat{Q}_i$ ,  $i = 1, 2, 3$ , are abelian if and only if  $b = 0$ . Due to the invariant way these groups were constructed, we conclude that  $G_{1,0}$  and  $G_{1,b}$  are not isomorphic.  $\square$

## REFERENCES

- [1] R. H. Bruck, A survey of binary systems. Springer-Verlag. 1958.
- [2] O. Chein, Moufang loops of small order, I, *Trans. Am. Math. Soc.* **188**, (1974), 31–51.
- [3] O. Chein, A. Rajah, Possible orders of nonassociative Moufang loops, *Comment. Math. Univ. Carolin.*, **41**, N 2 (2000), 237–244.
- [4] S. Doro, Simple Moufang loops, *Math. Proc. Camb. Phil. Soc.*, **83**, (1978), 377–392.
- [5] S. M. Gagola III, Hall’s theorem for Moufang loops. *J. Algebra*, **323**, N 12 (2010), 3252–3262.
- [6] A. N. Grishkov, A. V. Zavarnitsine, Lagrange’s theorem for Moufang loops, *Math. Proc. Camb. Phil. Soc.*, **139**, N 1 (2005), 41–57.
- [7] A. N. Grishkov, A. V. Zavarnitsine, Groups with triality, *J. Algebra Appl.*, **5**, N 4 (2006), 441–463.
- [8] A. N. Grishkov, A. V. Zavarnitsine, Sylow’s theorem for Moufang loops, *J. Algebra*, **321**, N 7 (2009), 1813–1825.
- [9] M. W. Liebeck, The classification of finite simple Moufang loops, *Math. Proc. Camb. Phil. Soc.*, **102**, (1987), 33–47.
- [10] A. Rajah, Moufang loops of odd order  $pq^3$ , *J. Algebra*, **235**, N 1 (2001), 66–93.